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Hopf- Bifurcation Analysis of Delayed Prey-Predator System

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ABSTRACT: The present paper deals with a SIS (Susceptible Infected Susceptible) predator-prey model with disease in prey species only. It is assumed that the predator species only predates the infected prey with Holling type - II functional response. It is assumed that predator growth is not instantaneous after consuming the prey and a discrete time lag for gestation of predator is required. The conditions for Hopf bifurcation around the interior equilibrium point are also derived. Finally, numerical simulations supporting the theoretical results are given.

Key Words: Local stability, Carrying capacity, Hopf bifurcation, Predation rate, Delay

INTRODUCTION

The existence of periodic solution / limit cycle oscillations due to predator functional response is well studied behavior in prey-predator models (Freedman, 1980 and Kot, 2001). A lot of work has already been done on different types of functional responses depending upon the densities of prey, predator and other significant factors (May, 2001, Murray, 2002, Agnihotri and Juneja, 2015, Agnihotri and Juneja, 2015). These functional responses are mostly classified as classical prey dependent and ratio dependent responses. Presently, the researchers are showing immense and continuing curiosity in studying the dynamics of predator–prey systems with time delay, stage structure, functional response, etc. In recent years, a number of researchers (Wang, 1998 and Huang, *et al.,* 2006) studied delay induced prey–predator model to discuss the stability of the system. Several investigations reported that under the influence of these factors, the system exhibits more complex and richer dynamics such as stability, bifurcation periodic solutions, persistence, etc. (Wang and Pei, 2011, Pallav, 2014, Wangersky and Cunningham, 1957, Meng, 2011 and Yu, 2009). It is quiet natural to believe that the conversion of prey biomass to predator biomass do not occur immediately; in fact, there is some time lag required for gestation. So we cannot obtain a sensible model without including time delays. It is observed that animals don't digest their food instantaneously which in turns delays their further activities.

Martin and Ruan 2001, Kar and Ghorai 2011 explored the effects of time lag in a prey–predator model incorporating parasite infection for the prey population. They concluded that the introduction of time delay in gestation period of predator has both stabilizing and destabilizing effects on the positive steady state.

Agnihotri and Juneja proposed a SIS predator prey model with disease in prey species. It is shown that the recovery of infected prey species plays a vital role in eliminating the limit cycle oscillations. The delay term has not been introduced in their paper. Keeping in mind this verity, an attempt has been made to form more realistic and interesting models of the population interactions by considering the time delay in the gestation period of predator population. The effect of time delay on the dynamics of the system and conditions for the existence of Hopf Bifurcation by considering time delay as the bifurcation parameter are also obtained.

The main objective of the present research is to form a more realistic model by incorporating the delay term in the gestation period of predator. The paper is structured as mathematical model, stability analysis of non zero equilibrium point, the conditions for the existence of Hopf bifurcation around non zero equilibrium point, numerical simulations are carried out to demonstrate the accuracy of the theoretical results. Finally gives a brief conclusion followed by acknowledgements.

FORMULATION OF MODEL

Let us consider prey predator model with time delay and having disease in prey species only. Holling Type-II functional response is considered for predator species. The delay term occurs in the interaction term or functional response term in the predator equation. It is considered that the reproduction of predators after predating the prey population is not immediate; thus it will be incorporated by some time lag τ required for the gestation of predators. Accordingly the delayed prey-predator model is as follows

$$\frac{dx}{dt} = r - \beta xy - \mu_1 x + \gamma y$$

$$\frac{dy}{dt} = \beta xy - \frac{yz}{1+y} - \mu_2 y - \gamma y \qquad (2.1)$$

$$\frac{dz}{dt} = z\left(-\mu_3 + \frac{cy(t-\tau)}{1+y(t-\tau)}\right)$$

where 'r' is the constant recruitment rate into the prey population, μ_1, μ_2 and μ_3 are the natural death rates of the susceptible prey, infected prey and the predator population respectively, 'γ' is the recovery rate of infected prey, 'β' is the incidence rate of disease in prey and 'c' is the conversion efficiency of predator.
The variables $x(t)$ and $y(t)$ denotes the respective densities of susceptible and infected prey at any time 't' and $z(t)$ denotes the predator population at any time 't'. Also all the parameters in the model takes positive values.

Suppose $x^{(0)}, y^{(0)}, z^{(0)}$ denotes the initial functions associated with system (2.1), then

$$\left\{\begin{array}{l} x^{(0)}(\theta) = \phi_1(\theta) \geq 0 \\ y^{(0)}(\theta) = \phi_2(\theta) \geq 0 \\ z^{(0)}(\theta) = \phi_3(\theta) \geq 0 \end{array}\right\} \quad \theta \in [-\tau, 0] ....(2.2)$$

$$\phi = (\phi_1, \phi_2, \phi_3) \in C\left([-\tau, 0], R^3_{+0}\right),$$

$$R^3_{+0} = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0\}$$

Positivity and Boundedness of the Solution

Lemma-1: All the solutions of the system (2.1) which initiate in R^3_{+0} are uniformly bounded.

Proof: Proof is obvious.

EXISTENCE AND STABILITY OF INTERIOR EQUILIBRIUM

Case-1:- $\tau = 0$

We are only interested with the endemic equilibrium point $\mathrm{E}\left(x^*, y^*, z^*\right)$ of the system (2.1) where all populations coexist.

$$\text{where } x^* = \frac{r(c-\mu_3) + \gamma\mu_3}{\mu_1(c-\mu_3) + \beta\mu_3}, \quad y^* = \frac{\mu_3}{c-\mu_3}, \quad z^* = \frac{c}{c-\mu_3}\left[\frac{(r\beta - \mu_1\mu_2 - \mu_1\gamma)(c-\mu_3) - \mu_2\mu_3\beta}{\mu_1(c-\mu_3) + \beta\mu_3}\right]$$

which exists provided

$$c > \mu_3, \ \gamma < \frac{1}{\mu_1}\left(r\beta - \mu_1\mu_2 - \frac{\mu_2\mu_3\beta}{c-\mu_3}\right) \qquad (3.1)$$

So if recovery rate becomes more and more low that equation (3.1) is satisfied, then the equilibrium point $\mathrm{E}\left(x^*, y^*, z^*\right)$ exists.

Case-2:- $\tau \neq 0$

The Characteristic roots corresponding to the equilibrium $E\left(x^*, y^*, z^*\right)$ are given by the expansion of determinant

$$\begin{vmatrix} \lambda+\beta y+\mu_1 & \beta x-\gamma & 0 \\ -\beta y & \lambda-\dfrac{yz}{(1+y)^2} & \dfrac{y}{1+y} \\ 0 & -\dfrac{cze^{-\tau\lambda}}{(1+y)^2} & \lambda \end{vmatrix} = 0$$

The characteristic equation becomes

$$\lambda^3 + A\lambda^2 + B\lambda = -e^{-\tau\lambda}\left(L\lambda + M\right) \quad (3.2)$$

$$A = \beta y + \mu_1 - \frac{yz}{(1+y)^2}, \quad B = \beta y(\beta x - \gamma) - (\beta y + \mu_1)\frac{yz}{(1+y)^2}$$

$$and\; L = \frac{cyz}{(1+y)^3}, \; M = \frac{cyz}{(1+y)^3}(\beta y + \mu_1)$$

For stability of $E\left(x^*, y^*, z^*\right)$, all the eigenvalues of characteristic equation (3.2) should have negative real parts. It is hard to found the conditions under which the equation (3.2) has all roots with negative real parts. Therefore the technique of stability change has been adopted to confer the stability and Hopf bifurcation of the system (2.1). In the following analysis, the delay τ is considered as the bifurcation parameter to study the local stability analysis and the Hopf bifurcation of the system. It is observed that small delay can stabilize the otherwise unstable equilibrium point, leading to Hopf Bifurcation. The possibility of Hopf Bifurcation is discussed in next section.

HOPF BIFURCATION ANALYSIS

For Hopf Bifurcation, there must exist a critical time delay τ_{cr} such that following two conditions are satisfied:

H1. $\lambda_{1,2}(\tau_{cr}) = \pm i\omega(\omega > 0)$ and all other eigenvalues have negative real part at $\tau = \tau_{cr}$

H2. $\mathrm{Re}\left[\dfrac{d\lambda_{1,2}(\tau_{cr})}{d\tau}\right] \neq 0$

The characteristic equation at positive equilibrium $E\left(x^*, y^*, z^*\right)$ is

$$\lambda^3 + A\lambda^2 + B\lambda = -e^{-\tau\lambda}\left(L\lambda + M\right) \quad (4.1)$$

For condition (H1) to hold, it is considered that a pair of imaginary roots exists for the equation (4.1) *i.e* $\lambda = i\omega(\omega > 0)$. On separating real and imaginary parts, following equations are obtained.

$$-M\cos\omega\tau - L\omega\sin\omega\tau = -A\omega^2$$

$$M\sin\omega\tau - L\omega\cos\omega\tau = -\omega^3 + B\omega$$

Squaring and adding the above equations, we get

Taking $\omega^2 = v$ we get the following cubic equation

$$v^3 + Q_1 v^2 + Q_2 v + Q_3 = 0 \quad (4.2)$$

where $Q_1 = A^2 - 2B,\; Q_2 = B^2 - L^2,\; Q_3 = -M^2 < 0$

The equation (4.2) must have at least one positive root. So H1 is satisfied.

$$\omega^6 + \left(A^2 - 2B\right)\omega^4 + \left(B^2 - L^2\right)\omega^2 - M^2 = 0$$

$$\text{Consequently } \tau_{cr} = \frac{1}{\omega_{cr}}\left[\cos^{-1}\left(\frac{\left(MA + L\omega_{cr}^2 - LB\right)\omega_{cr}^2}{M^2 + \omega_{cr}^2}\right) + 2k\pi\right]$$

$k = 0, 1, 2, \ldots\ldots\ldots$ (4.3)

Lemma-4.1:- Let $\lambda(\tau_{cr}) = i\omega_{cr}$ and $\gamma_{cr} = \omega_{cr}^2$ such that $\psi(\gamma_{cr}) = 0$, $\psi'(\gamma_{cr}) \neq 0$, then $\mathrm{Re}\left[\dfrac{d\lambda(\tau_{cr})}{d\tau}\right] \neq 0$ and its sign is same as that of $\psi'(\gamma_{cr})$.

Proof: - Again on differentiating Eq. (4.1) and rearranging the terms, we get

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{\left(3\lambda^2 + 2A\lambda + B\right)e^{\tau\lambda}}{\lambda\left(L\lambda + M\right)} + \frac{L}{\lambda\left(L\lambda + M\right)} - \frac{\tau}{\lambda} \tag{4.4}$$

$$\text{Now } \lambda\left(L\lambda + M\right)\Big|_{\tau=\tau_{cr}} = -L\omega_{cr}^2 + iM\,\omega_{cr} \tag{4.5}$$

$$\text{Now } \left(3\lambda^2 + 2A\lambda + B\right)e^{\tau\lambda}\Big|_{\tau=\tau_{cr}} = \left((-3\omega_{cr}^2 + B)\cos\omega_{cr}\tau_{cr} - 2A\omega\sin\omega_{cr}\tau_{cr}\right) + i\left((-3\omega_{cr}^2 + B)\sin\omega_{cr}\tau_{cr} + 2A\omega\cos\omega_{cr}\tau_{cr}\right) \tag{4.6}$$

Using (4.5) and (4.6) in (4.4) and noting that $\sigma = \left(L\omega_{cr}^2\right)^2 + \left(M\,\omega_{cr}\right)^2$

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_{cr}} = \mathrm{Re}\left[\frac{\left((-3\omega_{cr}^2 + B)\cos\omega_{cr}\tau_{cr} - 2A\omega\sin\omega_{cr}\tau_{cr}\right) + i\left((-3\omega_{cr}^2 + B)\sin\omega_{cr}\tau_{cr} + 2A\omega\cos\omega_{cr}\tau_{cr}\right)}{-L\omega_{cr}^2 + iM\,\omega_{cr}}\right] + \mathrm{Re}\left[\frac{L}{-L\omega_{cr}^2 + iM\,\omega_{cr}}\right]$$

Rationalizing and little simplification yields

$$\mathrm{Re}\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau_{cr}} = \frac{1}{\sigma}\left[3\omega_{cr}^6 + \left(2A^2 - 4B\right)\omega_{cr}^4 + \left(B^2 - L^2\right)\omega_{cr}^2\right]$$

$$= \frac{\omega_{cr}^2}{\sigma}\left[3\omega_{cr}^4 + (2A^2 - 4B)\omega_{cr}^2 + (B^2 - L^2)\right] = \frac{v}{\sigma}\psi'(v)$$

Theorem :- Hopf bifurcation exists for the system (2.1) with respect to the positive equilibrium point if $\psi'\left(\gamma_{cr}\right) \neq 0$ for $v_{\mathrm{cr}} = \omega_{cr}^2$.

NUMERICAL SIMULATIONS

To explore the Hopf bifurcation in the system numerically, the system is simulated in the absence of delay by considering the following set of values:

$$r = 11,\ \beta = 1,\ d_1 = 1,\ d_2 = 2.43,\ d_3 = 0.5,\ c = 1,\ \gamma = 0.03$$

It is found that the system exhibits limit cycle oscillations for the above set of values as shown in Fig. 1. Now we will show that even small delay in the gestation period can stabilize the system and thus remove the limit cycle oscillations from the system. So keeping all the parametric values same as above and taking $\tau = 0.91$, the system exhibits stable behavior as shown in Fig. 2.

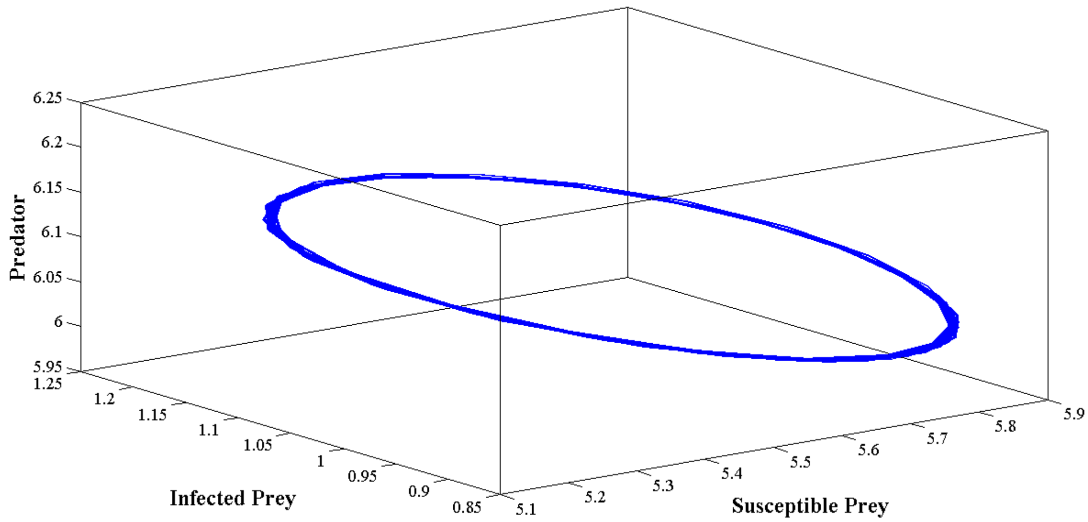

Fig. 1. Phase diagram depicting the limit cycle oscillations in absence of delay.

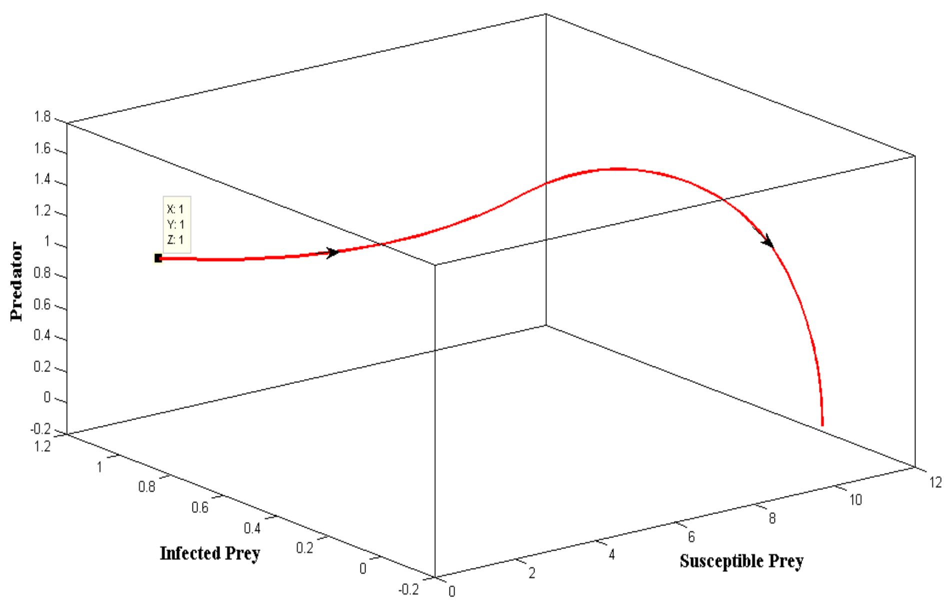

Fig. 2. Phase diagram showing the stability of interior equilibrium at $\tau = 0.91$.

CONCLUSION

An eco-epidemiological mathematical model incorporating time delay with infection in prey species is formed. It is found that the time delay can remove the limit cycle oscillations from the system. It is found that coexistence of all the three species is possible through periodic solutions due to Hopf bifurcation. Numerical simulations have been carried out to defend the theoretical results obtained. The numerical simulations have revealed that the dynamics of the system is largely affected by considering delay in the gestation period of the predator species.

The introduction of small delay in gestation period can make the system oscillation free and thus showing the stabilizing nature of delay.

FUTURE SCOPE

More eco-epidemiological models can be formed by taking infection in both the prey and predator species. The dynamics of the system can be studied by taking different delay terms in recovery of the species, gestation period of the predator etc. Effect of harvesting of both the prey and predator species on the dynamics of the system can also be studied.

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